

## SOME REMARKS ON VOLUME AND DIAMETER OF RIEMANNIAN MANIFOLDS

ROBERT BROOKS

In this note, we provide some remarks concerning a recent paper of Burger and Schroeder [4]. Their paper gives a relation between volume, diameter and the first eigenvalue of the Laplacian for compact quotients of rank 1 symmetric spaces. Here we will show how their results lead to analogous results for coverings of a fixed, but arbitrary, Riemannian manifold.

Theorem 2 of [4] states:

**Theorem ([4]).** Let  $\mathbf{H} = \mathbf{H}_{\mathbf{R}}^n$  for  $n \geq 4$ ,  $\mathbf{H}_{\mathbf{C}}^n$ ,  $\mathbf{H}_{\mathbf{H}}^n$ , or  $\mathbf{H}_{\mathbf{O}}^2$ .

Then there are constants  $a_n, b_n$  depending only on  $n$  such that for  $M$  a compact quotient of  $\mathbf{H}$ ,

$$\lambda_1(M) \leq \frac{a_n + b_n \log(\text{vol}(M))}{\text{diam}(M)}.$$

Note that for  $\mathbf{H} = \mathbf{H}_{\mathbf{R}}^n$  or  $\mathbf{H}_{\mathbf{C}}^n$  we may have  $\lambda_1(M)$  arbitrarily small. The fact that this is not the case for  $\mathbf{H} = \mathbf{H}_{\mathbf{H}}^n$ , or  $\mathbf{H}_{\mathbf{O}}^2$  follows from Kazhdan's Property T [5]. The fact that the isometry groups of these symmetric spaces have Property T is due to Kostant [11].

Our main result here is:

**Theorem 1.** Let  $M$  be an arbitrary compact manifold, and  $M_i$  a family of finite coverings of  $M$ . If there exists  $C > 0$  such that  $\lambda_1(M_i) > C$ , then there exist positive constants  $a, b$ , and  $c$  such that

$$a < \frac{\log(\text{vol}(M_i)) + c}{\text{diam}(M_i)} < b.$$

*Proof.* We first observe that, according to [7], for each  $n$ , and in particular for  $n = 4$ , there exists a compact quotient

$$N \text{ of } \mathbf{H}_{\mathbf{R}}^n$$

with a surjection  $\pi_1(N) \rightarrow \mathbb{Z} * \mathbb{Z}$ .

Now suppose that  $\pi_1(M)$  is generated by  $k$  elements. Then there is a finite covering  $N'$  of  $N$  and a surjection  $\phi: \pi_1(N') \rightarrow \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ times}} \rightarrow \pi_1(M)$ .

Let  $N'_i$  be the coverings of  $N'$  induced from those of  $M_i$ —

$$\pi_1(N'_i) = \phi^{-1}(\pi_1(M_i)).$$

*Claim.* There exist constants  $C'$ ,  $d'$  and  $k'$  such that

- (a)  $\lambda_1(N'_i) > C'$ ,
- (b)  $\text{vol}(N'_i) = d' \text{vol}(M_i)$ ,
- (c)  $\text{diam}(N'_i) > k' \text{diam}(M_i)$ .

*Proof.* (a) is just Theorem 4 of [3].

(b) follows with  $d' = \text{vol}(N')/\text{vol}(N)$

(c) follows from the Milnor-Svarc lemma [8], which implies that  $\text{diam}(N'_i)$  and  $\text{diam}(M_i)$  are both estimated up to constants by the group-theoretic diameter of  $\pi_1(M)/\pi_1(M_i)$ , relative to a fixed set of generators for  $\pi_1(M)$ .

We now apply the theorem of [4] to show that there is a constant  $a'$  with

$$a' < \frac{\log(\text{vol}(N'_i)) + c}{\text{diam}(N'_i)}.$$

It follows from the Claim that

$$\begin{aligned} \frac{\log(\text{vol}(M_i)) + \log(d') + c}{\text{diam}(M_i)} &\geq \frac{\log(\text{vol}(N'_i)) + c}{k' \text{diam}(N'_i)} \\ &\geq (\text{const}) \frac{\log(\text{vol}(N'_i)) + c}{\text{diam}(N'_i)} \geq (\text{const}) a'. \end{aligned}$$

The inequality

$$\frac{\log(\text{vol}(M_i))}{\text{diam}(M_i)} \leq b$$

is true in complete generality, and follows immediately from the Comparison Theorem.  $b$  depends only on a lower bound for the curvature of  $M$  and the dimension of  $M$ . Combining these gives Theorem 1.

We remark here that one could also prove Theorem 1 by use of a graph theoretic isoperimetric inequality due to Alon and Milman [1], see also Gromov-Milman [12]. It is worth remarking that the ideas that go into the

proof of [1] (which is completely elementary) are similar in many points to the ideas behind the proof of [4].

We may extend the ideas in our proof of Theorem 1 to show:

**Theorem 2.** *For each  $n$ , there exists a compact hyperbolic  $n$ -manifold  $N$  and coverings  $N_i$  of  $N$  such that*

- (i)  $\lambda_1(N_i) \rightarrow 0$  as  $i \rightarrow \infty$ .
- (ii) *There exists  $C > 0$  such that  $\log(\text{vol}(N_i))/\text{diam}(N_i) > C$ .*

*Proof.* Let us begin with an arbitrary manifold  $M$  with a family of coverings  $M_i$  such that  $\lambda_1(M_i)$  is bounded away from 0 and  $\text{diam}(M_i) \rightarrow \infty$ . For instance, we could choose  $M$  with  $\pi_1(M) = SL(2, \mathbb{Z})$ , and  $M_i$  the congruence coverings of  $M$ .

Now let  $M' = M \times S^1$ , and for each  $k$  let  $M'_k$  be the covering of  $M'$  whose fundamental group is  $\pi_1(M_i) \oplus ([\log(\text{diam}(M_i))] \times \mathbb{Z}) \subset \pi_1(M) \oplus \mathbb{Z}$ .

To see that

$$\lambda_1(M'_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

we compute the isoperimetric constant  $h(M'_k)$ . But dividing  $M'_k$  into two pieces along the fibers of antipodal points of the  $[\log(\text{diam}(M_i))]$ -fold cover of  $S^1$ , shows that

$$h(M'_k) \leq \frac{2 \text{vol}(M_k)}{1/2[\log(\text{diam}(M_k))] \text{vol}(M_k)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

The fact that  $\lambda_1(M'_k) \rightarrow 0$  as  $k \rightarrow \infty$  then follows from Theorem 1 of [3], or can be seen directly. But  $\text{vol}(M'_k) = \text{vol}(M_k) \times [\log(\text{diam}(M_k))]$  and  $\text{diam}(M'_k) \leq (\text{const})(\text{diam}(M_k) + [\log(\text{diam}(M_k))])$  as can be seen again from the Milnor-Svarc lemma.

Hence,

$$\frac{\log(\text{vol}(M'_k))}{\text{diam}(M'_k)} \geq \frac{\log[\log(\text{diam}(M_k))] + \log \text{vol}(M_k)}{2 \text{const}(\text{diam}(M_k))} \geq \text{const},$$

since

$$\frac{\log(\text{vol}(M_k))}{\text{diam}(M_k)} \geq \text{const by Theorem 1.}$$

We now repeat the argument of Theorem 1 to find a hyperbolic manifold  $N$  with a surjective map  $\pi_1(N) \rightarrow \pi_1(M \times S^1)$ , whose coverings have the same properties.

As an example of this circle of ideas, we show:

**Theorem 3.** For each  $n \geq 2$ , let us choose generators for  $SL(n, \mathbb{Z})$ . Then there is a constant  $C_n$  depending only on  $n$  and the choice of generators, such that  $\text{diam}(SL(n, \mathbb{Z}/p)) < C_n \log p$ .

*Proof.* We first observe that if  $\pi_1(M) = SL(n, \mathbb{Z})$ , the congruence coverings  $M^p$  of  $M$  satisfy  $\lambda_1(M^p) > C$  for some  $C > 0$ .

When  $n \geq 3$ , this follows from the fact that  $SL(n, \mathbb{R})$  has Property T and [3]. When  $n = 2$  this follows from [3] and Selberg's Theorem [9] that  $\lambda_1(\mathbb{H}^2/\Gamma_p) \geq 3/16$ , where  $\Gamma_p$  is the  $p$ th congruence subgroup. The fact that [3] applies despite the noncompactness of  $\mathbb{H}^2/SL(2, \mathbb{Z})$  is discussed in [2].

It follows from Theorem 1 that

$$\frac{\log(\text{vol}(SL(n, \mathbb{Z}/p)))}{\text{diam}(SL(n, \mathbb{Z}/p))} \geq a$$

for some  $a \geq 0$ . But  $\log(\text{vol}(SL(n, \mathbb{Z}/p))) \leq \log(p^{n^2}) = C_n \cdot \log(p)$  and the theorem is proved.

**Corollary 4.** For  $p$  a prime number, consider the set  $V_p = \{0, 1, \dots, p-1, \infty\}$ . Then there is a  $C$  independent of  $p$  such that any  $a, b \in V_p$  can be joined by a sequence of at most  $C \log(p)$  moves of the type  $x \rightarrow x+1$ ,  $x-1$ ,  $x \rightarrow \bar{x}$ , where  $\bar{x}$  is the multiplicative inverse of  $x \pmod{p}$ ,  $\bar{0} = \infty$ , and  $\overline{\infty} = 0$ .

*Proof.* This is the graph of  $SL(2, \mathbb{Z})/\Gamma_p^*$ , where

$$\Gamma_p^* \supset \Gamma_p \text{ is the Hecke group } \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p} \right\}.$$

We close this paper with the following example, shown to us by John Millson and based on work of R. Livne [6]:

**Theorem 5.** There exists a compact quotient  $M$  of  $\mathbb{H}_\mathbb{C}^2$ , such that  $\pi_1(M)$  surjects onto  $\mathbb{Z} * \mathbb{Z}$ .

At present, we don't have examples of  $\mathbb{H}_\mathbb{C}^n$ ,  $n > 2$ , whose fundamental group surjects onto  $\mathbb{Z} * \mathbb{Z}$ .

We remark that Theorem 5 allows us to extend those results of [4] (in particular remark (iii)) and the present paper which only applied to  $\mathbb{H}_\mathbb{R}^n$  also to  $\mathbb{H}_\mathbb{C}^2$ .

*Proof.* We consider the following situation: for each  $N$ , let  $X(N)$  be the compactified moduli space for elliptic curves with level  $N$  structure, and  $E(N)$  the universal elliptic curve of level  $N$ . Then when  $N \geq 3$ ,  $X(N)$  is a smooth Riemann surface, and there is a submersion  $E(N) \rightarrow X(N)$  which, away from finitely many points of  $X(N)$ , is a fibration whose fibers are elliptic curves.

There are  $N^2$  sections of this fibration taking a point in  $X(N)$  to one of the  $N^2$  points of order  $N$  on the fiber.

For each integer  $d$ , let  $S_d(N)_\Delta$  be a  $d$ -fold cyclic branched covering of  $E(N)$  which is totally ramified along these  $N^2$  sections, and is a covering away from these sections.

By calculating Chern numbers, and appealing to a characterization of quotients of  $\mathbf{H}_\mathbb{C}^2$  due to Yau [10], Livne showed in [6] that  $S_d(N)_\Delta$  can be realized as a compact quotient of  $\mathbf{H}_\mathbb{C}^2$  precisely when  $(N, d)$  is one of the pairs  $(7, 7)$ ,  $(8, 4)$ ,  $(9, 3)$  or  $(12, 2)$ . For these values, he also explicitly constructs a realization of  $\pi_1(S_d(N)_\Delta)$  as a discrete, cocompact subgroup of  $PU(2, 1)$ . In all of these cases,  $X$  is a Riemann surface of genus  $\geq 2$ .

We now claim:

*Claim:*  $\pi_1(S_d(N)_\Delta)$  surjects onto  $\mathbb{Z} * \mathbb{Z}$ .

*Proof.* It suffices to show that  $\pi_1(S_d(N)_\Delta)$  surjects onto  $\pi_1(X(N))$ .

So pick a base-point  $p$  in  $S_d(N)_\Delta$ , and a point  $p_E$  in  $E(N)$  lying over  $p$  and not a point of order  $N$ .

If  $\gamma$  is any loop at  $p$ , we jiggle it slightly if necessary to guarantee that  $\gamma$  avoids the singular values on  $X(N)$ . We then use the fact that  $E(N) \rightarrow X(N)$  is a fibration away from the singular values to lift  $\gamma$  to a curve on  $E(N)$  starting at  $p_E$ . Since the fibers are connected, we may close this curve up to a loop  $\tilde{\gamma}$  based at  $p_E$  which projects onto  $\gamma$ .

We may now jiggle  $\tilde{\gamma}$  so that it avoids the  $N^2$  sections, and so lift it to a curve on  $S_d(N)_\Delta$ . Again since the fibers are connected, we may close this up to a closed curve on  $S_d(N)_\Delta$  whose homotopy class projects onto that of  $\gamma$ , showing that  $\pi_1(S_d(N)_\Delta)$  surjects onto  $\pi_1(X(N))$ . This completes the claim, and hence the theorem.

## References

- [1] N. Alon & V. D. Milman,  $\lambda_1$ , *isoperimetric inequalities for graphs, and superconcentrators*, J. Combinatorial Theory Ser. B **38** (1985) 73–88.
- [2] R. Brooks, *On the angles between certain arithmetically defined subspaces of  $\mathbb{C}^n$* , Ann. Inst. Fourier (Grenoble), to appear.
- [3] ———, *The spectral geometry of a tower of coverings*, J. Differential Geometry **23** (1986) 97–107.
- [4] M. Burger & V. Schroeder, *Volume, diameter, and the first eigenvalue of locally symmetric spaces of rank one*, J. Differential Geometry **26** (1987) 273–284.
- [5] D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Functional Anal. Appl. **1** (1968) 63–65.
- [6] R. Livne, *On certain covers of the universal elliptic curve*, Ph.D. thesis, Harvard University, November, 1981.
- [7] J. Millson, *On the first betti number of a constant negatively curved manifold*, Ann. of Math. (2) **104** (1976) 235–247.

- [8] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry **2** (1968) 1–7.
- [9] A. Selberg, *On the estimation of fourier coefficients of modular forms*, Proc. Sympos. Pure Math., Vol. 8, Amer. Math. Soc., Providence, RI, 1965, 1–15.
- [10] S. T. Yau, *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977) 1798–1799.
- [11] B. Kostant, *On the existence and irreducibility of certain series of representations*, Lie Groups and their Representations, (I. M. Gelfand, ed.), Halsted, NY, 1975.
- [12] M. Gromov & V. D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105** (1983) 843–854.

UNIVERSITY OF SOUTHERN CALIFORNIA